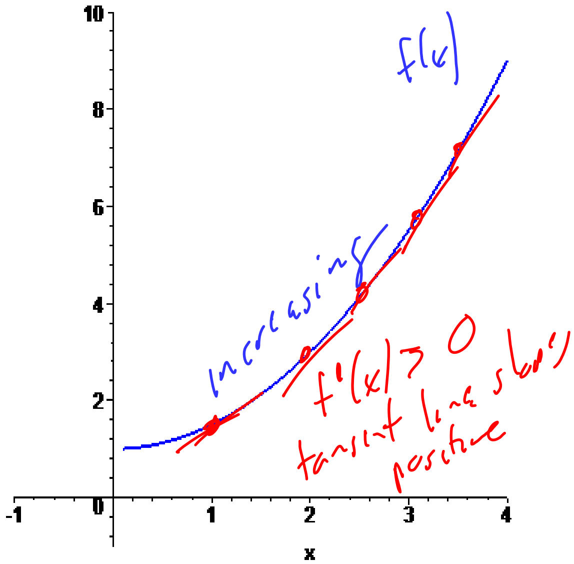
Increasing and Decreasing Functions and The First Derivative Test

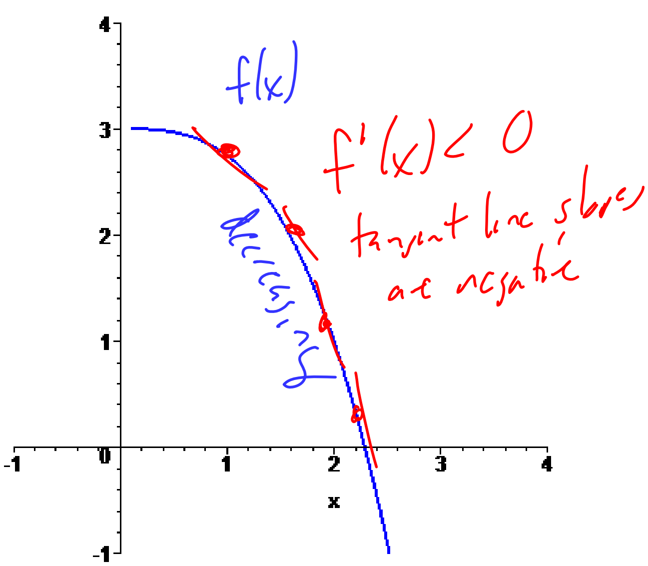
Concavity and The Second Derivative Test

Practice HW

Increasing and Decreasing Test for Functions

Given a function f defined on an interval.
1. If $f'(x) > 0$ on an interval, then f is increasing on that interval.
2. If $f'(x) < 0$ on an interval, then f is decreasing on that interval.





Example 2: State whether the function $f(x) = x^3 - 3x$ is increasing, decreasing, or neither on an interval "near" $x = 2$.

Solution: $f(x) = x^3 - 3x$

$f'(x) = 3x^2 - 3$

At $x = 2$ $\rightarrow$ $f'(2) = 3(2)^2 - 3 = 9 > 0$ $f(x)$ is increasing at $x = 2$!

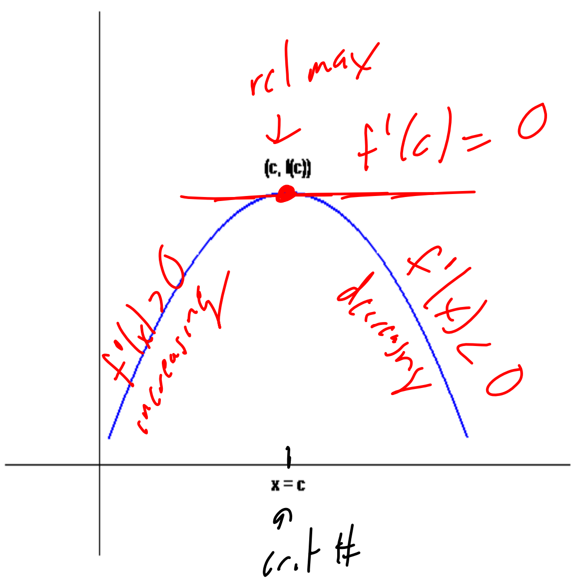
At $x = 0$ $\rightarrow$ $f'(0) = 3(0)^2 - 3 = -3 < 0$ $f(x)$ is decreasing at $x = 2$!

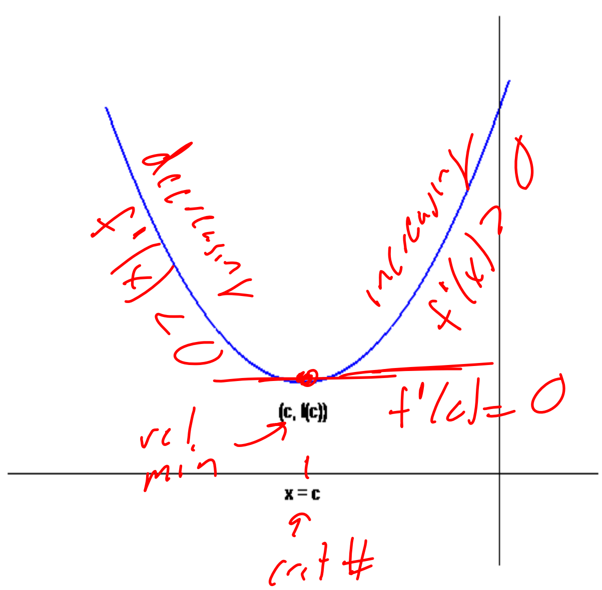
First Derivative Test

$f'(c) = 0$ or $f'(c)$ is undefined

Suppose $x = c$ is a critical number of a continuous function f.

1. If $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$, $f(c)$ is a relative maximum.
2. If $f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c$, $f(c)$ is a relative minimum.

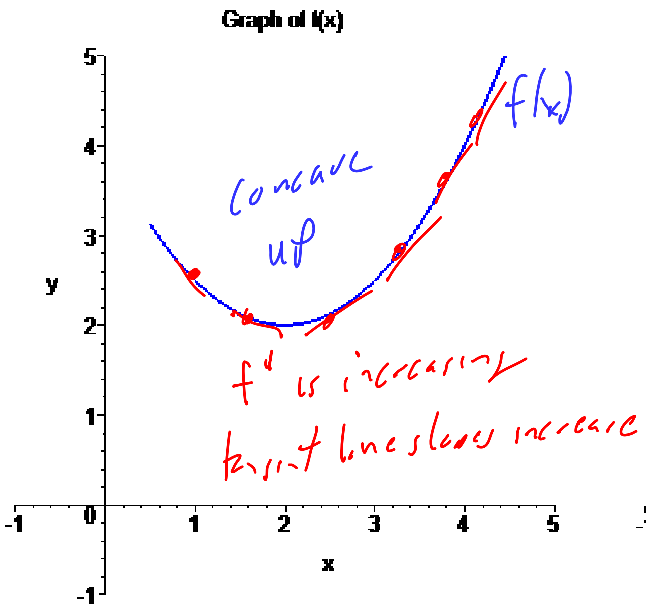


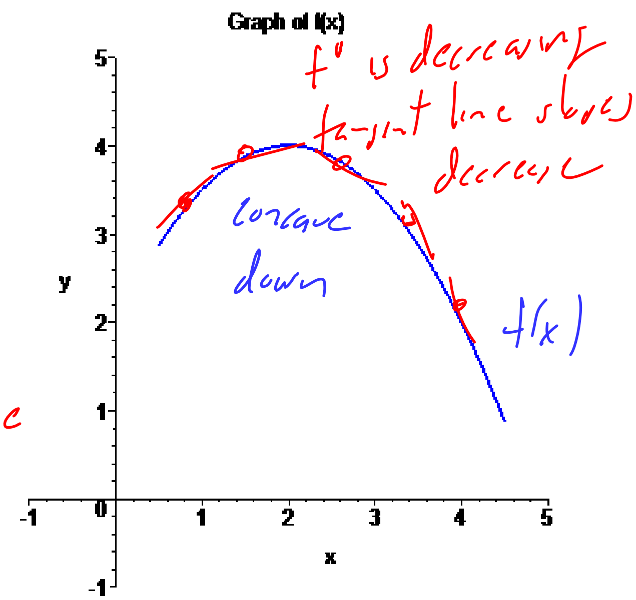


Concavity

- A function f is <u>concave</u> <u>up</u> on an interval I if f' is increasing on I.

- A function f is <u>concave</u> <u>down</u> on an interval I if f' is decreasing on I.





Concavity Test for Functions

Given a function f defined on an interval.
1. If $f''(x) > 0$ on an interval, then f is <u>concave</u> <u>up</u> on that interval.
2. If $f''(x) < 0$ on an interval, then f is <u>concave</u> <u>down</u> on that interval.

Example 2: State whether the function $f(x) = x^3 - 3x$ is concave up, concave down, or neither on an interval "near" x = -1.

Solution:

$$f(x) = x^3 - 3x$$
$$f'(x) = 3x^2 - 3$$
$$f''(x) = 6x$$

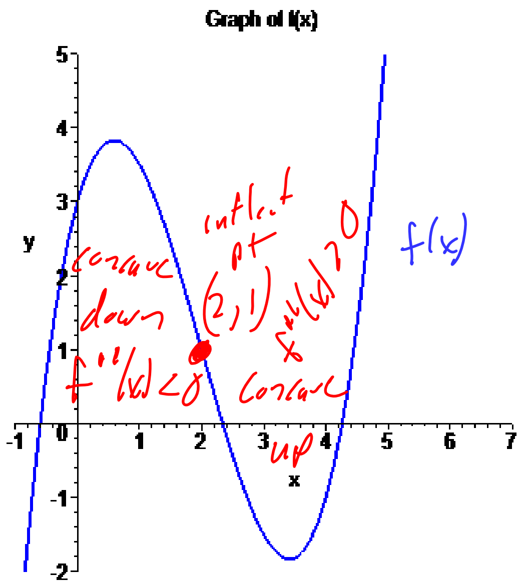
At $x = -1$ $\rightarrow$ $f''(-1) = 6(-1) = -6 < 0$ $f(x)$ is concave down at $x = -1$

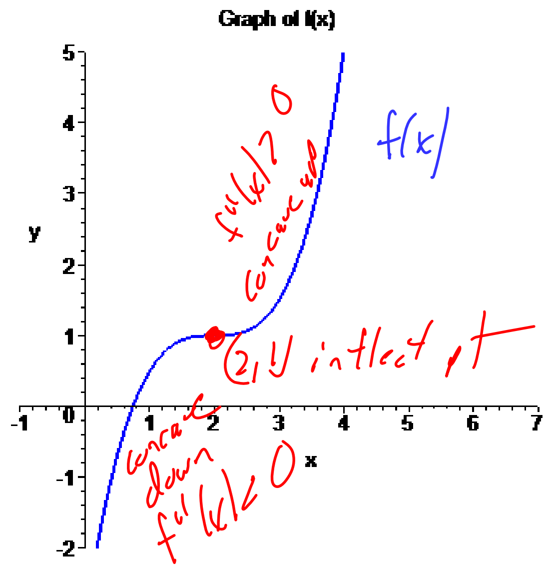
■

At $x = 2$ $\rightarrow$ $f''(2) = 6(2) = 12 > 0$ $f(x)$ is concave up at $x = 2$

Inflection Points

Inflection points are points where the concavity of the graph of a function f changes.

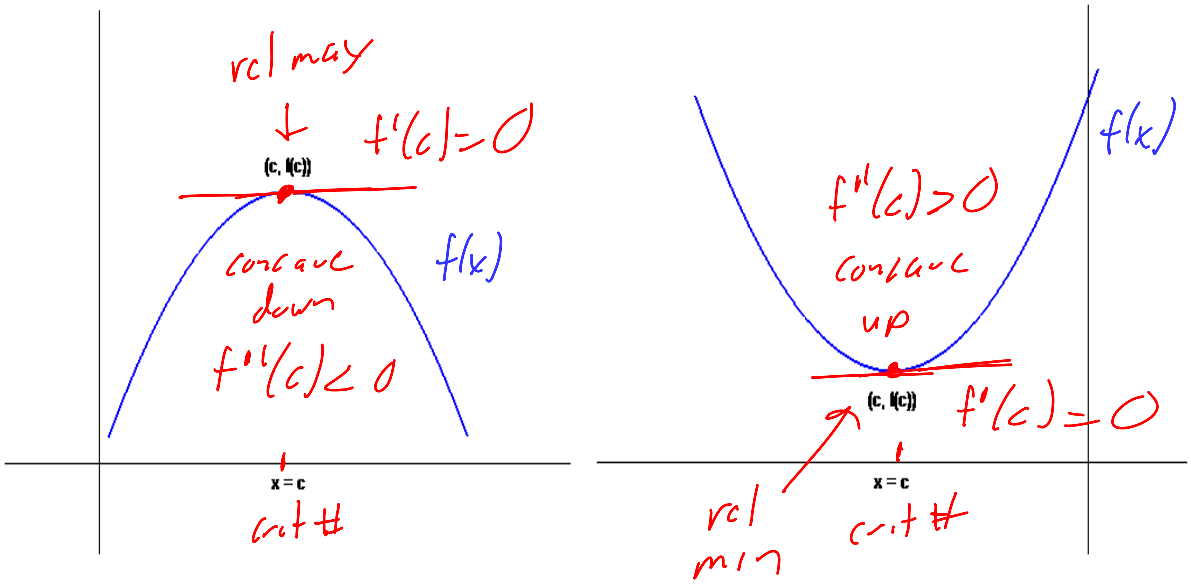




Note: If $(c, f(c))$ is a point of inflection of the graph of f, then either $f''(c) = 0$ or $f''(c)$ is undefined.

Second Derivative Test (Test For Relative Maximum and Relative Minimum Points)

Let f be a function where $x = c$ is a critical point where $f'(c) = 0$.
1. If $f''(c) > 0$, then $f(c)$ is a relative minimum.
2. If $f''(c) < 0$, then $f(c)$ is a relative maximum.
3. If $f''(c) = 0$ or $f''(c)$ is undefined, the test fails – use the 1st derivative test.



Procedure For Graphing Functions Using Derivatives

Given a function $f(x)$.

1. State the domain of the function.
2. Compute f' and f''.
3. Find the critical numbers, which are values of x where either $f'(x)=0$ or $f'(x)$ is undefined. Find the y coordinates of these points by substituting these x values back into the original function $f(x)$. These points represent the candidates for the local maximum and minimum points.
4. Use the 2nd derivative test to determine whether critical numbers are local Maximum or local minimum points. If the 2nd derivative test fails, use the 1st derivative test with a sign diagram, testing whether the 1st derivative is positive or negative to the left and right of the critical numbers.
5. Look for candidates for the points of inflection by finding values of x where either $f''(x)=0$ or $f''(x)$ is undefined. Find the y coordinates of these points by substituting these x values back into the original function $f(x)$. These points represent the candidates for the points of inflection.
6. Test the inflection point candidates by testing the concavity to the left and the right of inflection point candidates using a sign diagram with the 2nd derivative.
7. Use the information to sketch the graph.

Example 3: For the function $f(x) = -2x^2 + 8x + 3$, determine the intervals where the function is increasing and decreasing, local maximum and local minimum points, the intervals where the function is convave up and convave down, and the points of inflection. Use the information to sketch the graph.

Solution: $f(x) = -2x^2 + 8x + 3$

$f'(x) = -4x + 8$, $f''(x) = -4$

Crit #'s: set $f'(x) = -4x + 8 = 0$

$-4x + 8 = 0$

$-4x = -8$

Crit #: $x = \frac{-8}{-4} = 2$

Note: $f'(x)$ is defined for a x, no undefined values for crit #'s

Find y coord for crit #

$y = f(x) = -2x^2 + 8x + 3$

$x = 2 \rightarrow y = f(2) = -2(2)^2 + 8(2) + 3$

$= -8 + 16 + 3 = 11$

pt: $(2, 11)$

$(2, 11)$ is a candidate for a rel max/min pt

Use 2nd Der test to test rel max/min candidate

Cand

$f''(x) = -4$

$(2, 11) \rightarrow f''(2) = -4 < 0$ concave down rel max

$x = 2$

pts of inflection: set $f''(x) = -4 = 0$

$-4 = 0$ impossible $(-4 \neq 0)$

no points of inflection!

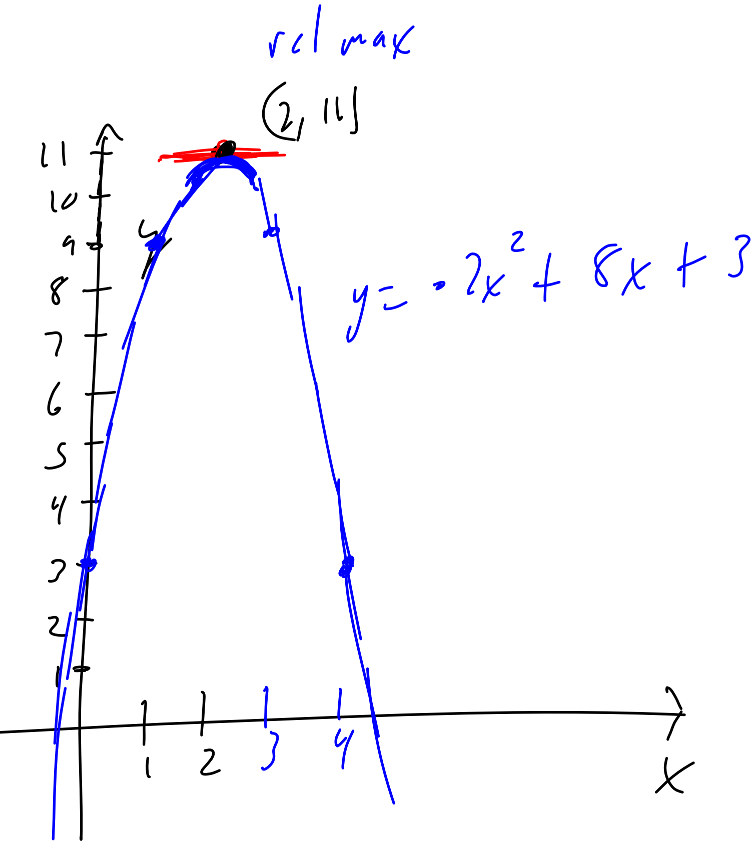
Note! since $f''(x) = -4 < 0$ for x, the graph of $f(x)$ is always concave down!

Note: To get a more accurate graph, plot pts around rel max/min pts

x	$y = f(x) = -2x^2 + 8x + 3$
0	3
1	$-2(1)^2 + 8(1) + 3 = 9$
2	11 rel max
3	9
4	3

Intervals	Interval not
Increasing: $x < 2$	$(-\infty, 2)$
Decreasing: $x > 2$	$(2, \infty)$
Always concave down	$(-\infty, \infty)$

Never concave up!



Example 4: For the function $f(x) = 2x^3 + 21x^2 - 108x + 8$, determine the intervals where the function is increasing and decreasing, local maximum and local minimum points, the intervals where the function is convave up and convave down, and the points of inflection. Use the information to sketch the graph.

Solution: $f(x) = 2x^3 + 21x^2 - 108x + 8$

$f'(x) = 6x^2 + 42x - 108$, $f''(x) = 12x + 42$

crit #'s: set $f'(x) = 6x^2 + 42x - 108 = 0$

$$\frac{\cancel{6}(x^2 + 7x - 18)}{\cancel{6}} = \frac{0}{6}$$

$x^2 + 7x - 18 = 0$

$(x+9)(x-2) = 0$

$x + 9 = 0$, $x - 2 = 0$

crit #'s: $x = -9$, $x = 2$

Find y coord's for crit #'s

$y = f(x) = 2x^3 + 21x^2 - 108x + 8$

$x = -9 \to y = f(-9) = 2(-9)^3 + 21(-9)^2 - 108(-9) + 8$
$= 1223$

pt $(-9, 1223)$

$x = 2 \to y = f(2) = 2(2)^3 + 21(2)^2 - 108(2) + 8$
$= -108$

pt $(2, -108)$

pts $(-9, 1223)$ and $(2, -108)$

rel max/min candidates

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use 2nd Derivative Test to test candidates

Cand $\quad f''(x) = 12x + 42$

$(-9, 1223) \to x = -9 \to f''(-9) = 12(-9) + 42 = -66 < 0$ concave down, rel max

$(2, -108) \to x = 2 \to f''(2) = 12(2) + 42 = 66 > 0$ concave up, rel min inflection pt: set $f''(x) = 12x + 42 = 0$

$12x + 42 = 0$

$12x = -42$

$x = \frac{-42}{12} = -3.5$

y coord

$y = f(x) = 2x^3 + 21x^2 - 108x + 8$

$y = f(-3.5) = 2(-3.5)^3 + 21(-3.5)^2 - 108(-3.5) + 8$

$= 557.5$

pt $(-3.5, 557.5)$

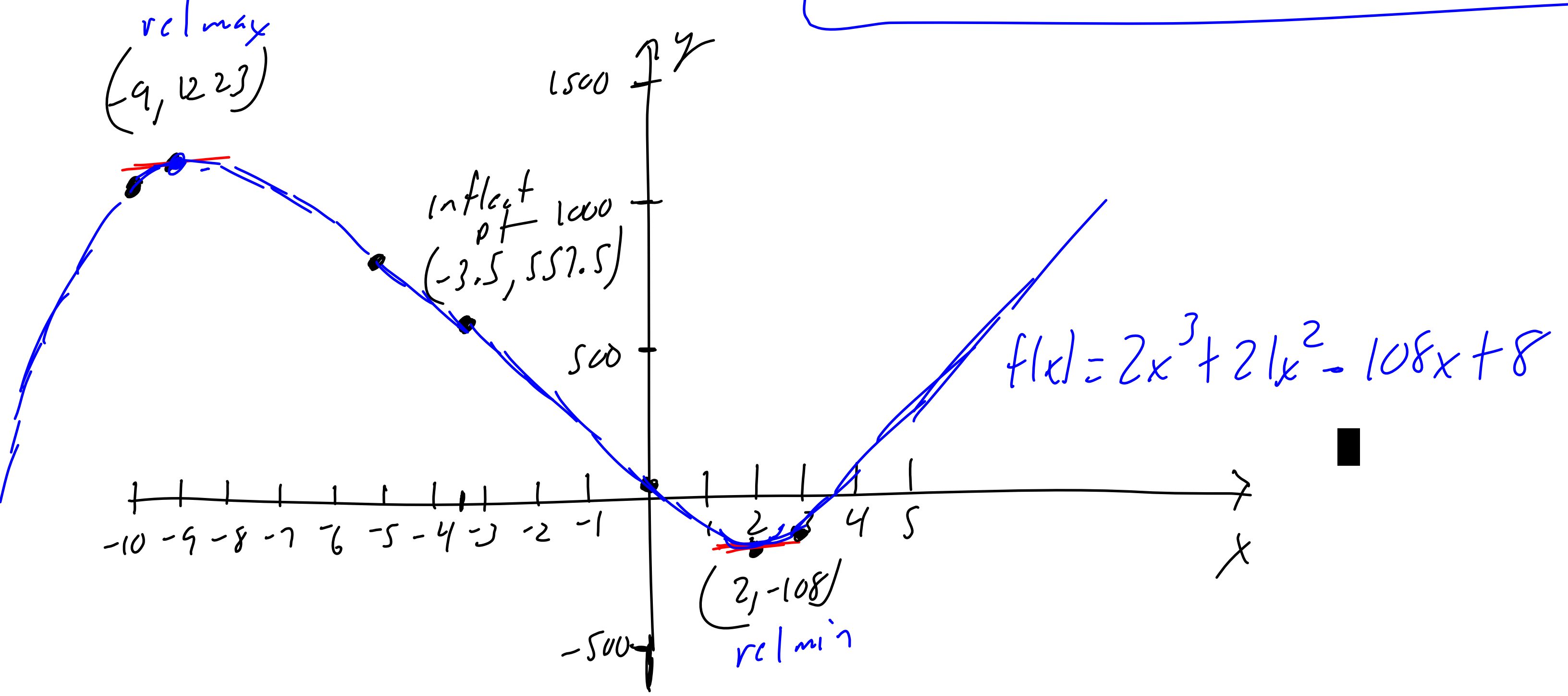
inflection pt candidate: $(-3.5, 557.5)$

Graph changes from concave down to up is an inflection pt

To make graph more accurate, plot pts

x	$y = f(x) = 2x^3 + 21x^2 - 108x + 8$
-10	$2(-10)^3 + 21(-10)^2 - 108(-10) + 8 = 1188$
-9	1223 rel max
-5	823
-3.5	557.5 inflect pt
0	8
2	-108 rel min
3	-73

	Intervals	Interval Not
Increasing:	$x < -9,\ x > 2$	$(-\infty, -9) \cup (2, \infty)$
Decreasing:	$-9 < x < 2$	$(-9, 2)$
Concave up:	$x > 3.5$	$(3.5, \infty)$
Concave down:	$x < -3.5$	$(-\infty, 3.5)$



Example 5: For the function $f(x) = x^4 - 4x^3$, determine the intervals where the function is increasing and decreasing, local maximum and local minimum points, the intervals where the function is convave up and convave down, and the points of inflection. Use the information to sketch the graph.

$f(x) = x^4 - 4x^3$

$f'(x) = 4x^3 - 12x^2, \quad f''(x) = 12x^2 - 24x$

crit #'s

set $f'(x) = 4x^3 - 12x^2 = 0$

$4x^3 - 12x^2 = 0$

$4x^2(x-3) = 0$

$4x^2 = 0, \quad x - 3 = 0$

$x^2 = \frac{0}{4}, \quad x - 3 = 0$

$x^2 = 0$

$\underbrace{x = 0, \quad x = 3}_{\text{crit #'s}}$

Find y coord's of crit #'s

$y = f(x) = x^4 - 4x^3$

$x = 0 \Rightarrow y = f(0) = (0)^4 - 4(0)^3$
$= 0 - 0 = 0$

pt $(0,0)$

$x = 3 \Rightarrow y = f(3) = (3)^4 - 4(3)^3$
$= 81 - 108$
$= -27$

pt $(3,-27)$

pts $(0,0)$ and $(3,-27)$ are local max min candidates

Test candidates using 2nd Derivative Test

$f''(x) = 12x^2 - 24x$

$(0,0) \Rightarrow x = 0 \Rightarrow f''(0) = 12(0)^2 - 24(0) = 0$ Test Fails! Use 1st Der test

$(3,-27) \Rightarrow x = 3 \Rightarrow f''(3) = 12(3)^2 - 24(3) = 36 > 0$ Concave up $\cup$ rel min

Use 1st Der test - apply a sign diagram using $f'(x)$ to the left and right of crit #'s to find out where graph is increasing + decreasing

$f'(x) = 4x^3 - 12x^2$

decreasing $f'(x) < 0$ — crit # $x=0$ — decreasing $f'(x) < 0$ — crit # $x=3$ — increasing $f'(x) > 0$

$- - - - - - - - - - | - - - - - - - - | + + + + + + + + + +$ x

pick $x = -1 < 0$

$f'(-1) = 4(-1)^3 - 12(-1)^2$

$= -4 - 12$

$= -16 < 0$

decreasing

$x < 0$

pick $x = 1$

$0 < 1 < 3$

$f'(1) = 4(1)^3 - 12(1)^2$

$= 4 - 12$

$= -8 < 0$

decreasing

$0 < x < 3$

pick $x = 4 > 3$

$f'(4) = 4(4)^3 - 12(4)^2$

$= 256 - 192$

$= 64 > 0$

increasing

$x > 3$

Note: Graph decreases $x < 0$, and also decreases $0 < x < 3$, pt $(0,0)$ is neither a rel max or rel min

Graph decreases $0 < x < 3$ and increases $x > 3$, graph changes from decreasing to increasing $(3,-27)$ a rel min

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pts of inflection

set $f''(x) = 12x^2 - 24x = 0$

$12x^2 - 24x = 0$

$12x(x-2) = 0$

$12x = 0, \quad x - 2 = 0$

$x = \frac{0}{12} = 0, \quad x = 2$

Find y coords

$y = f(x) = x^4 - 4x^3$

$x = 0 \Rightarrow y = f(0) = (0)^4 - 4(0)^3 = 0$

pt $(0,0)$

$x = 2 \Rightarrow y = f(2) = (2)^4 - 4(2)^3$

$= 16 - 32$

$= -16$

pt $(2,-16)$

$(0,0)$ $(2,-16)$ are inflection pt candidates

To test the inflect pt cand, use a sign diag on second derivative to see where concavity changes

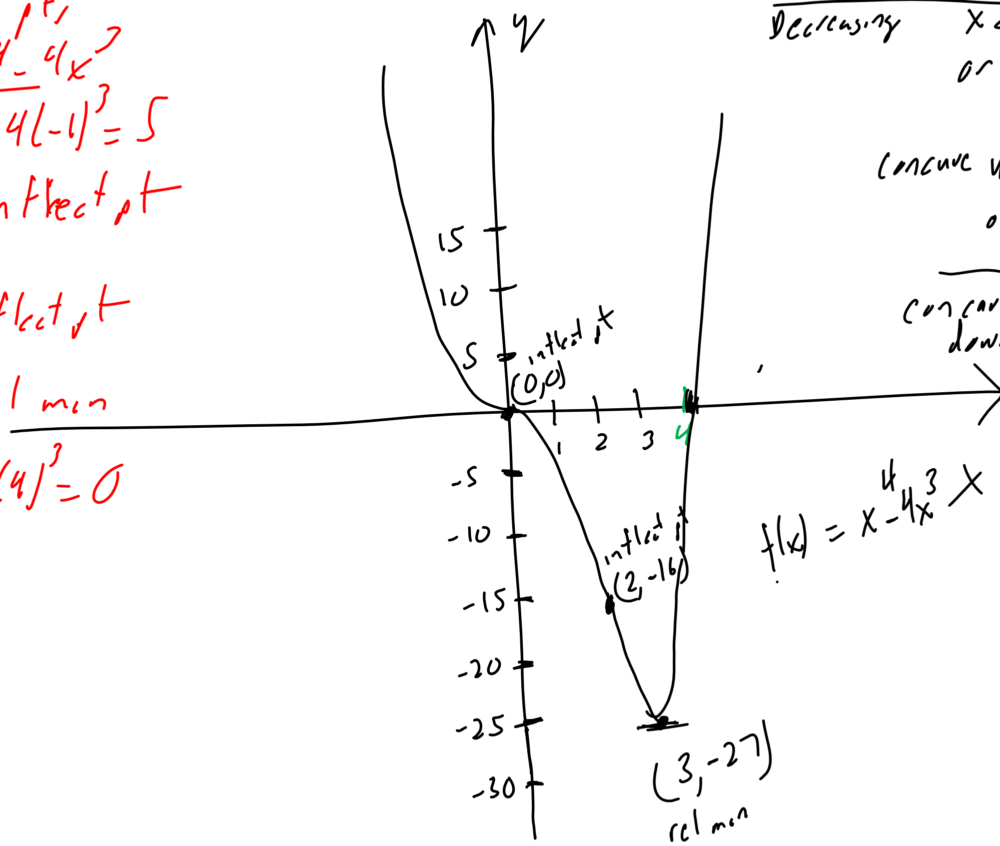
$f''(x) = 12x^2 - 24x$

concave up		concave down		concave up
$f''(x) > 0$		$f''(x) < 0$		$f''(x) > 0$
+ + + + + + + + + +	0 inflect cand	− − − − − − −	2 inflect cand	+ + + + + + + + +
pick $x = -1 < 0$		pick $x = 1$, $0 < 1 < 2$		pick $x = 3 > 2$
$f''(-1) = 12(-1)^2 - 24(-1)$		$f''(1) = 12(1)^2 - 24(1)$		$f''(3) = 12(3)^2 - 24(3)$
$= 12 + 24$		$= -12 < 0$		$= 108 - 72$
$= 36 > 0$		concave down		$= 36 > 0$
concave up		$0 < x < 2$		concave up
$x < 0$				$x > 2$

Concavity changes at both $x = 0$, $x = 2$. Hence $(0,0)$ and $(2,-16)$ are pts of inflection

plot extra pts

x	$y = x^4 - 4x^3$
-1	$(-1)^4 - 4(-1)^3 = 5$
0	0 inflect pt
2	-16 inflect pt
3	-27 rel min
4	$(4)^4 - 4(4)^3 = 0$



Increasing: $x > 3$ or $(3, \infty)$

Decreasing: $x < 0$, $0 < x < 3$ or $(-\infty, 0) \cup (0, 3)$

concave up: $x < 0$, $x > 2$ or $(-\infty, 2) \cup (2, \infty)$

concave down: $0 < x < 2$. or $(0, 2)$

Example 6: For the function $f(x) = 3x^{2/3} - x$, determine the intervals where the function is increasing and decreasing, local maximum and local minimum points, the intervals where the function is convave up and convave down, and the points of inflection.
Use the information to sketch the graph.

Solution: Note that for this function $f(x) = 3x^{2/3} - x$, there is no values of x where the $f(x)$ is undefined. Hence, the domain of this function is all real numbers or $(-\infty, \infty)$. We next compute the first and second derivatives. We see that

$$f'(x) = 3(\frac{2}{3})x^{-1/3} - 1 = 2x^{-1/3} - 1 = \frac{2}{x^{1/3}} - 1 = \frac{2}{\sqrt[3]{x}} - 1.$$

Using the form $f'(x) = 2x^{-1/3} - 1$ for the first derivative, we see that

$$f''(x) = 2(-\frac{1}{3})x^{-4/3} - 0 = -\frac{2}{3x^{4/3}} = \frac{-2}{3\sqrt[3]{x^4}}.$$

Hence, $f'(x) = \frac{2}{\sqrt[3]{x}} - 1$ and $f''(x) = \frac{-2}{3\sqrt[3]{x^4}}$.

We next find the critical numbers. Note that in this case, $f'(x) = \frac{2}{\sqrt[3]{x}} - 1$ is undefined at $x = 0$. Therefore, $x = 0$ is one critical number. We next look for critical numbers where $f'(x) = 0$. This gives

$$f'(x) = \frac{2}{\sqrt[3]{x}} - 1 = 0$$

$$\frac{2}{\sqrt[3]{x}} = 1 \qquad \text{(add 1 to both sides)}$$

$$\sqrt[3]{x} \cdot \frac{2}{\sqrt[3]{x}} = \sqrt[3]{x} \cdot 1 \quad \text{(Multiply both sides by } \sqrt[3]{x}\text{)}$$

$$\sqrt[3]{x} = 2 \qquad \text{(Rearrange and simplify0}$$

$$(\sqrt[3]{x})^3 = (2)^3 \quad \text{(Eliminate the radical by raising both sides to the 3}^{\text{rd}}\text{ power)}$$

$$x = 8 \qquad \text{(Simplify and solve for } x\text{)}$$

Hence, the critical numbers are $x = 0$ and $x = 8$. **(continued on next page)**

For the critical numbers, we find their corresponding *y* coordinates. To do this, we must use the original function $f(x) = 3x^{2/3} - x$. This gives the following.

$x = 0 \Rightarrow y = f(0) = 3(0)^{2/3} - 0 = 0 - 0 = 0$ gives the point (0, 0)

$x = 8 \Rightarrow y = f(8) = 3(8)^{2/3} - 8 = 3[8^{\frac{1}{3}}]^2 - 8 = 3[\sqrt[3]{8}]^2 - 8 = 3(2)^2 - 8 = 3(4) - 8 = 12 - 8 = 4$
gives the point (8, 4)

Hence, (0, 0) and (8, 4) are the local maximum and minimum point candidates.
To test these points, we first try the 2nd derivative test using $f''(x) = \dfrac{-2}{3\sqrt[3]{x^4}}$. This gives the following information for each candidate.

$(0,0) \Rightarrow x = 0 \Rightarrow f''(0) = \dfrac{-2}{3\sqrt[3]{0^4}} = -\dfrac{2}{0}$ which is undefined. The test fails.

$(8,4) \Rightarrow x = 8 \Rightarrow f''(8) = \dfrac{-2}{3\sqrt[3]{8^4}} = -\dfrac{2}{3\sqrt[3]{4096}} = -\dfrac{2}{3(16)} = -\dfrac{1}{24} < 0$. This says the graph is concave down at this point or that (8, 4) is a local maximum.

Hence, the 2nd derivative test says (8, 4) is a local maximum but is inconclusive for the point (0, 0). So we need more information. We get it by using the 1st derivative test by testing the sign of the 1st derivative to the left and right of each critical number (which of course are the same as the *x* coordinates of the local maximum and minimum candidates).

1st Derivative Test Table for $f'(x) = \dfrac{2}{\sqrt[3]{x}} - 1$

$f'(x) < 0$ − − − − − − − − − Decreasing	**Crit Num *x* = 0**	$f'(x) > 0$ + + + + + + + + + Increasing	**Crit Num *x* = 8**	$f'(x) < 0$ − − − − − − − − − Decreasing
Choose *x* = -1 < 0 $f'(-1) = \dfrac{2}{\sqrt[3]{-1}} - 1$ $= \dfrac{2}{-1} - 1$ $= -2 - 1$ $= -3 < 0$ Decreasing		**Choose *x* = 1 Note 0 < 1 < 8** $f'(1) = \dfrac{2}{\sqrt[3]{1}} - 1$ $= \dfrac{2}{1} - 1$ $= 2 - 1$ $= 1 > 0$ Increasing		**Choose *x* = 9 > 8** $f'(1) = \dfrac{2}{\sqrt[3]{9}} - 1$ $= \dfrac{2}{9^{1/3}} - 1$ $\approx \dfrac{2}{2.08} - 1$ $= 0.96 - 1$ $= -0.04 < 0$ Decreasing

(continued on next page)

To the left of the critical number $x = 0$, the 1st derivative test table says the graph of $f(x)$ is decreasing. To the right of $x = 0$, the 1st derivative test table says the graph of $f(x)$ is increasing. By definition, this says the point (0, 0) is a local minimum.

To the left of the critical number $x = 8$, the 1st derivative test table says the graph of $f(x)$ is increasing. To the right of $x = 8$, the 1st derivative test table says the graph of $f(x)$ is decreasing. By definition, this says the point (8, 4) is a local maximum (which agrees with the assertion already established by the 2nd Derivative test).

The table also gives information where the graph of $f(x)$ is increasing and decreasing. Summarizing, we see that

$f(x)$ is increasing when $0 < x < 8$ or in interval notation (0, 8)
$f(x)$ is decreasing when $x < 0, x > 8$ or in interval notation $(-\infty, 0) \cup (8, \infty)$

Now, we look for points of inflection. Recall that points of inflection occur at points where the second derivative $f''(x) = 0$ or where $f''(x)$ is undefined. Above, we computed the second derivative $f''(x) = \dfrac{-2}{3\sqrt[3]{x^4}}$. Note that $f''(x) = \dfrac{-2}{3\sqrt[3]{x^4}}$ is undefined at $x = 0$. Thus $x = 0$ is inflection point candidate. If we set

$$f''(x) = \frac{-2}{3\sqrt[3]{x^4}} = 0$$

$$3\sqrt[3]{x^4} \cdot \frac{-2}{3\sqrt[3]{x^4}} = 3\sqrt[3]{x^4} \cdot 0 \qquad \text{(Multiply both sides by } 3\sqrt[3]{x^4} \text{ to eliminate the fraction)}$$

$$-2 = 0 \qquad \text{(Simplify)}$$

Since $-2 \neq 0$, this produces not solution. Thus, $x = 0$ is the only inflection point candidate. Recall that its y-coordinate on the graph is obtain by substituting into the original function $f(x) = 3x^{2/3} - x$. Recall that this gives the following.

$x = 0 \Rightarrow y = f(0) = 3(0)^{2/3} - 0 = 0 - 0 = 0$ gives the point (0, 0)

To test whether the concavity changes, we use a test table involving the second derivative.

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2nd Derivative Test Table for $f''(x) = \dfrac{-2}{3\sqrt[3]{x^4}}$

$f''(x) < 0$ – – – – – – – – – – Concave Down	Inflection point Candidate ***x* = 0**	$f''(x) < 0$ – – – – – – – – – – Concave Down
Choose ***x* = -1 < 0** $f''(-1) = \dfrac{-2}{3\sqrt[3]{(-1)^4}}$ $= \dfrac{-2}{3\sqrt[3]{1}}$ $= \dfrac{-2}{3(1)}$ $= -\dfrac{2}{3} < 0$ Concave Down		**Choose** ***x* = 1 > 0** $f''(1) = \dfrac{-2}{3\sqrt[3]{(1)^4}}$ $= \dfrac{-2}{3\sqrt[3]{1}}$ $= \dfrac{-2}{3(1)}$ $= -\dfrac{2}{3} < 0$ Concave Down

Since the concavity of the graph does not change to the left and right of $x = 0$ (it remains concave down), the point (0, 0) does not produce a point of inflection. Since there are no other candidates, the graph of $f(x)$ has to points of inflection. Using the table, we see that we have the following intervals for concavity.

Concave Down**:** When $x < 0$, $x > 0$ or in interval notation $(-\infty, 0) \cup (0, \infty)$.
Concave Up: Never

In summary, this is the information we have produced:

1. The point (0, 0) is a local minimum, the point (8, 4) is a local maximum.
2. $f(x)$ is increasing on the interval (0, 8).
   $f(x)$ is decreasing on the interval $(-\infty, 0) \cup (8, \infty)$.
3. There are no points of inflection.
4. The graph is concave down on the interval $(-\infty, 0) \cup (0, \infty)$.
   The graph is never concave up.
5. Note that the local minimum, (0, 0), was obtained from an undefined critical point, that is, $f'(x) = \dfrac{2}{\sqrt[3]{x}} - 1$ was undefined at $x = 0$. A function has an undefined derivative at a point where the graph is not continuous, has a sharp point, or a vertical tangent. The function $f(x) = 3x^{2/3} - x$ is continuous at $x = 0$. Since (0, 0) is a local minimum, the graph changes from decreasing to increasing at this point. In addition, the concavity remains down to the left and right of this point. Hence, the graph produces a sharp point at (0, 0).

(continued on next page)

The following is the graph of the function $f(x) = 3x^{2/3} - x$.